Bock & Bargmann

Case I: The Quasi-Simplex

For repeated measurements (learning trials—the example variables involve scores at stages of learning on a two-hand coordination task):

"According to the simplex model, each of these variables incorporates a new component of skill at that stage of practice. These components are assumed to combine additively to determine the score of each subject at the respective stage of practice."

Bock & Bargmann, p. 523

Algebraically, that gives a model for the test scores

$$\boldsymbol{y}_i = \boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{\xi}_i + \boldsymbol{\epsilon}_i$$

for person i, in which $\boldsymbol{\xi}_i$ is a latent variable and $\boldsymbol{A}$ is fixed and known:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

$$\boldsymbol{y}_i = \boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{\xi}_i + \boldsymbol{\epsilon}_i$$

implies that the observed variables as distributed in multivariate normal form with mean $\boldsymbol{\mu}$ and covariance matrix:

$$\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{\Phi}\boldsymbol{A}' + \boldsymbol{\Gamma}$$

Case I, the only one we'll discuss, restricts the latent variables to be uncorrelated,

$$\boldsymbol{\Phi} = \text{diag}[\phi_1, \phi_2, \dots, \phi_m]$$

and the error variances to be homoscedastic:

$$\boldsymbol{\Gamma} = \gamma \boldsymbol{I}$$

The multivariate normal likelihood is:

$$L = \prod_{i=1}^{N} \frac{|\boldsymbol{\Sigma}|^{-1/2}}{(2\pi)^{p/2}} \exp[-\frac{1}{2}(\boldsymbol{y}_i - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_i - \boldsymbol{\mu})]$$

The maximum likelihood estimate of $\boldsymbol{\mu}$ is (either) obviously (or see Anderson, 1958, p. 47) the mean vector $\boldsymbol{y}_.$ If we let the sum of products of the N observations corrected to the sample mean be

$$N \cdot \boldsymbol{S} = \sum_{i=1}^{N} \boldsymbol{y}\boldsymbol{y}' - N\boldsymbol{y}.\boldsymbol{y}.',$$

and the loglikelihood to be maximized to estimate the parameters that yield $\boldsymbol{\Sigma}$ is:

$$\log L = -\frac{Np}{2}\log 2\pi - \frac{N}{2}\log|\boldsymbol{\Sigma}| - \frac{N}{2}\text{tr}\boldsymbol{\Sigma}^{-1}\boldsymbol{S}.$$

Bock and Bargmann show how to figure the derivatives of that loglikelihood, and use a Newton-Raphson algorithm to find parameter estimates that maximize it.

Here we'll look at:

- Derivative-free R
- R with derivatives
- C++ (with derivatives; no choice here)