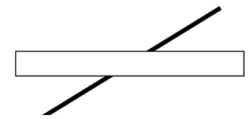
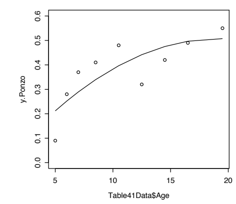
Bock's Chapter 4.1

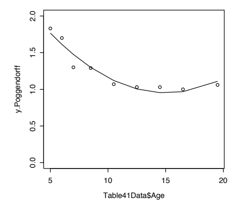
The Ponzo and Poggendorff Illusions and Age

Ponzo Illusion | Poggendorff Illusion

(Both figures are rotated relative to those in Bock's book.)







Bock's chapter 4, section 1, gives three (3) arguments for the least squares solution for regression coefficients—derivatives equal to zero, "complete the square," and a geometric presentation.

Consider any that are useful for you?

What follows is a likelihood-based argument:

The multivariate normal likelihood for the data is:

$$L = \frac{\frac{1}{\sigma}|\boldsymbol{D}|^{1/2}}{(2\pi)^{n/2}} \exp[-\frac{1}{2}(\boldsymbol{y}_. - \boldsymbol{X\beta})'\frac{1}{\sigma^2}\boldsymbol{D}(\boldsymbol{y}_. - \boldsymbol{X\beta})]$$

So the loglikelihood is proportional to:

$$\ell \sim -\frac{1}{2}\frac{1}{\sigma^2}(\boldsymbol{y}_. - \boldsymbol{X\beta})'\boldsymbol{D}(\boldsymbol{y}_. - \boldsymbol{X\beta})$$

$$\ell \sim -\frac{1}{2}\frac{1}{\sigma^2}(\boldsymbol{y}_.'\boldsymbol{D}\boldsymbol{y}_. - 2\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{D}\boldsymbol{y}_. + \boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{DX\beta})$$

And the derivative of the loglikelihood with respect to the parameters is [using matrix calculus rules such as on p. 41ff of Bock's "other" chapter 2, or Searle (1982)] is on the following slide:

$$\ell \sim -\frac{1}{2}\frac{1}{\sigma^2}(\boldsymbol{y}_.'\boldsymbol{D}\boldsymbol{y}_. - 2\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{D}\boldsymbol{y}_. + \boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{DX\beta})$$

The derivative with respect to the parameters is:

$$\frac{\partial \ell}{\partial \boldsymbol{\beta}} = \frac{1}{\sigma^2}(\boldsymbol{X}'\boldsymbol{D}\boldsymbol{y}_. - \boldsymbol{X}'\boldsymbol{DX\beta})$$

That has to be zero for the ML estimates of the coefficients:

$$\frac{\partial \ell}{\partial \boldsymbol{\beta}} = \frac{1}{\sigma^2}(\boldsymbol{X}'\boldsymbol{D}\boldsymbol{y}_. - \boldsymbol{X}'\boldsymbol{DX\beta}) = 0$$

So

$$\boldsymbol{X}'\boldsymbol{D}\boldsymbol{y}_. = \boldsymbol{X}'\boldsymbol{DX}\hat{\boldsymbol{\beta}}$$

and

$$(\boldsymbol{X}'\boldsymbol{DX})^{-1}\boldsymbol{X}'\boldsymbol{D}\boldsymbol{y}_. = \hat{\boldsymbol{\beta}}$$

Returning to the the first derivatives:

$$\frac{\partial \ell}{\partial \boldsymbol{\beta}} = \frac{1}{\sigma^2}(\boldsymbol{X}'\boldsymbol{D}\boldsymbol{y}_. - \boldsymbol{X}'\boldsymbol{DX\beta}) = 0$$

The second derivatives are:

$$\left(\frac{\partial^2 \ell}{\partial \boldsymbol{\beta}^2}\right) = -\frac{1}{\sigma^2}\boldsymbol{X}'\boldsymbol{DX}$$

The negative inverse of the [expected value (here that makes no difference) of the] matrix of second derivatives is the error covariance matrix of the estimates:

$$\mathcal{V}(\boldsymbol{\beta}) = \sigma^2(\boldsymbol{X}'\boldsymbol{DX})^{-1}$$

(All of this gives the same results Bock obtains without reference to likelihood.)